ON THE L^p-THEORY OF ANISOTROPIC SINGULAR PERTURBATIONS OF ELLIPTIC PROBLEMS

CHOKRI OGABI

Academie de Grenoble, 38300 France

Abstract. In this article we give an extention of the L^2-theory of anisotropic singular perturbations for elliptic problems. We study a linear and some non-linear problems involving L^p data $(1 < p < 2)$. Convergences in pseudo Sobolev spaces are proved for weak and entropy solutions, and rate of convergence is given in cylindrical domains

1. Introduction

1.1. **Preliminaries.** In this article we shall give an extension of the L^2-theory of the asymptotic behavior of elliptic, anisotropic singular perturbations problems. This kind of singular perturbations has been introduced by M. Chipot [6]. From the physical point of view, these problems can modelize diffusion phenomena when the diffusion coefficients in certain directions are going toward zero. The L^2 theory of the asymptotic behavior of these problems has been studied by M. Chipot and many co-authors. First of all, let us begin by a brief discussion on the uniqueness of the weak solution (by weak a solution we mean a solution in the sense of distributions) to the problem

$$\begin{cases} -\operatorname{div}(A\nabla u) = f \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded Lipschitz domain, we suppose that $f \in L^p(\Omega)$ $(1 < p < 2)$. The diffusion matrix $A = (a_{ij})$ is supposed to be bounded and satisfies the ellipticity assumption on Ω (see assumptions (2) and (3) in subsection 1.2). It is well known that (1) has at least a weak solution in $W_0^{1,p}(\Omega)$. Moreover, if A is symmetric and continuous and $\partial\Omega \in C^2$ [2] then (1) has a unique solution in $W_0^{1,p}(\Omega)$. If A is discontinuous the uniqueness assertion is false, in [15] Serrin has given a counterexample when $N \geq 3$. However, if $N = 2$ and if $\partial\Omega$ is sufficiently smooth and without any continuity assumption on A, (1) has a unique weak solution in $W_0^{1,p}(\Omega)$. The proof is based on the Meyers regularity theorem (see for instance [13]). To treat this pathology, Benilin, Boccardo, Gallouet, and al have introduced the concept of the entropy solution [4] for problems involving L^1 data (or more generally a Radon measure).

Date: 28,January 2015.

1991 *Mathematics Subject Classification.* 35J15, 35B60, 35B25.

Key words and phrases. Anisotropic singular perturbations, elliptic problem, L^p-theory, entropy solutions. Asymptotic behaviour. rate of convergence.

For every $k>0$ We define the function $T_k:\mathbb{R}\to\mathbb{R}$ by

$$T_k(s)=\left\{\begin{array}{cc} s & ,|s|\leq k \\ ksgn(x) & |s|\geq k \end{array}\right.$$

And we define the space $\mathcal{T}_0^{1,2}$ introduced in [4].

$$\mathcal{T}_0^{1,2}(\Omega)=\left\{\begin{array}{c} u\ :\Omega\to\mathbb{R}\text{ measurable such that for any }k>0\text{ there exists} \\ (\phi_n)\subset H_0^1(\Omega):\phi_n\to T_k(u)\text{ a.e in }\Omega \\ \text{and }(\nabla\phi_n)_{n\in\mathbb{N}}\text{ is bounded in }L^2(\Omega) \end{array}\right\}$$

This definition of $\mathcal{T}_0^{1,2}$ is equivalent to the original one given in [4].In fact, this is a characterization of this space [4]. Now, more generally, for $f\in L^1(\Omega)$ we have the following definition of entropy solution [4].

Definition 1. *A function $u\in\mathcal{T}_0^{1,2}(\Omega)$ is said to be an entropy solution to (1) if*

$$\int_\Omega A\nabla u\cdot\nabla T_k(u-\varphi)dx\leq\int_\Omega fT_k(u-\varphi)dx,\ \varphi\in\mathcal{D}(\Omega),\ k>0$$

We refer the reader to [4] for more details about the sense of this formulation. The main results of [4] show that (1) has a unique entropy solution which is also a weak solution of (1) moreover since Ω is bounded then this solution belongs to $\bigcap_{1\leq r<\frac{N}{N-1}} W_0^{1,r}(\Omega)$.

1.2. **Description of the problem and functional setting.** Throughout this article we will suppose that $f\in L^p(\Omega)$, $1<p<2$, (we can suppose that $f\notin L^2(\Omega)$). We give a description of the linear problem (some nonlinear problems will be studied later). Consider the following singular perturbations problem

$$\left\{\begin{array}{l} -\operatorname{div}(A_\epsilon\nabla u_\epsilon)=f \\ u_\epsilon=0\quad\text{on }\partial\Omega \end{array}\right. ,\tag{2}$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$. Let $q\in\mathbb{N}^*$, $N-q\geq 2$. We denote by $x=(x_1,...,x_N)=(X_1,X_2)\in\mathbb{R}^q\times\mathbb{R}^{N-q}$ i.e. we split the coordinates into two parts. With this notation we set

$$\nabla=(\partial_{x_1},...,\partial_{x_N})^T=\begin{pmatrix}\nabla_{X_1}\\ \nabla_{X_2}\end{pmatrix},$$

where

$$\nabla_{X_1}=(\partial_{x_1},...,\partial_{x_q})^T\text{ and }\nabla_{X_2}=(\partial_{x_{q+1}},...,\partial_{x_N})^T$$

Let $A=(a_{ij}(x))$ be a $N\times N$ matrix which satisfies the ellipticity assumption

$$\exists\lambda>0:A\xi\cdot\xi\geq\lambda\left|\xi\right|^2\ \ \forall\xi\in\mathbb{R}^N\text{ for a.e }x\in\Omega,\tag{3}$$

and

$$a_{ij}(x)\in L^\infty(\Omega),\forall i,j=1,2,....,N,\tag{4}$$

We have decomposed A into four blocks

$$A=\left(\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right),$$

where $A_{11},\ A_{22}$ are respectively $q \times q$ and $(N-q)\times(N-q)$ matrices. For $0<\epsilon\leq 1$ we have set

$$A_\epsilon = \begin{pmatrix} \epsilon^2 A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}$$

We denote $\Omega_{X_1} = \left\{X_2 \in \mathbb{R}^{N-q} : (X_1, X_2) \in \Omega\right\}$ and $\Omega^1 = P_1\Omega$ where $P_1 : \mathbb{R}^N \to \mathbb{R}^p$ is the usual projector. We introduce the space

$$V_p = \left\{ \begin{array}{l} u \in L^p(\Omega) \mid \nabla_{X_2} u \in L^p(\Omega), \\ \quad \text{and for a.e } X_1 \in \Omega^1, u(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1}) \end{array} \right\}$$

We equip V_p with the norm

$$\|u\|_{V_p} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla_{X_2} u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}},$$

then one can show easily that $(V_p, \|\cdot\|_{V_p})$ is a separable reflexive Banach space. The passage to the limit (formally) in (2) gives the limit problem

$$\left\{ \begin{array}{l} -\operatorname{div}_{X_2}(A_{22}\nabla_{X_2} u_0(X_1, \cdot)) = f\ (X_1, \cdot) \\ u_0(X_1, \cdot) = 0 \quad \text{on } \partial\Omega_{X_1} \qquad X_1 \in \Omega^1 \end{array} \right. \tag{5}$$

The L^2-theory (when $f \in L^2$) of problem (2) has been treated in [8], convergence has been proved in V_2 and rate of convergence in the L^2-norm has been given. For the L^2-theory of several nonlinear problems we refer the reader to [9],[10],[14]. This article is mainly devoted to study the L^p-theory of the asymptotic behavior of linear and nonlinear singularly perturbed problems. In other words, we shall study the convergence $u_\epsilon \to u_0$ inV_p (Notice that in [9], authors have treated some problems involving L^p data where some others data of the equations depend on p, one can check easily that it is not the L^p theory which we expose in this manuscript). Let us briefly summarize the content of the paper:

- In section 2: We study the linear problem, we prove convergences for weak and entropy solutions.
- In section 3: We give the rate of convergence in a cylindrical domain when the data is independent of X_1.
- In section 4: We treat some nonlinear problems.

2. The Linear Problem

The main results in this section are the following

Theorem 1. *Assume (3), (4) then there exists a sequence $(u_\epsilon)_{0<\epsilon\leq 1} \subset W_0^{1,p}(\Omega)$ of weak solutions to (2) and $u_0 \in V_p$ such that $\epsilon\nabla_{X_1} u_\epsilon \to 0$ in $L^p(\Omega)$, $u_\epsilon \to u_0$ in V_p where u_0 satisfies (5) for a.e $X_1 \in \Omega^1$.*

Corollary 1. *Assume (3), (4) then if A is symmetric and continuous and $\partial\Omega \in C^2$, then there exists a unique $u_0 \in V_p$ such that $u_0(X_1;\cdot)$ is the unique solution to (5) in $W_0^{1,p}(\Omega_{X_1})$ for a.e X_1. Moreover the sequence $(u_\epsilon)_{0<\epsilon\leq 1}$ of the unique solutions (in $W_0^{1,p}(\Omega)$) to (2) converges in V_p to u_0 and $\epsilon\nabla_{X_1} u_\epsilon \to 0$ in $L^p(\Omega)$.*

Proof. This corollary follows immediately from Theorem 1 and uniqueness of the solutions of (2) and (5) as mentioned in subsection 1.1 (Notice that $\partial\Omega_{X_1} \in C^2$). $\square$

Theorem 2. *Assume (3), (4) then there exists a unique $u_0 \in V_p$ such that $u_0(X_1,\cdot)$ is the unique entropy solution of (5). Moreover, the sequence of the entropy solutions $(u_\epsilon)_{0<\epsilon\leq 1}$ of (2) converges to u_0 in V_p and $\epsilon\nabla_{X_1}u_\epsilon \to 0$ in $L^p(\Omega)$.*

2.1. **Weak convergence.** Let us prove the following primary result

Theorem 3. *Assume (3), (4) then there exists a sequence $(u_{\epsilon_k})_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ of weak solutions to (2) ($\epsilon_k \to 0$ as $k\to\infty$) and $u_0 \in V_p$ such that $\nabla_{X_2}u_{\epsilon_k} \rightharpoonup \nabla_{X_2}u_0$, $\epsilon_k\nabla_{X_1}u^n_{\epsilon_k} \rightharpoonup 0$, $u_{\epsilon_k} \rightharpoonup u_0$ in $L^p(\Omega-weak.$ and u_0 satisfies (5) for a.e $X_1 \in \Omega^1$.*

Proof. By density let $(f_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ be a sequence such that $f_n \to f$ in $L^p(\Omega)$, we can suppose that $\forall n \in \mathbb{N} : \|f_n\|_{L^p} \leq M$, $M \geq 0$. Consider the regularized problem

$$u^n_\epsilon \in H^1_0(\Omega), \quad \int_\Omega A_\epsilon \nabla u^n_\epsilon \cdot \nabla\varphi dx = \int_\Omega f_n \varphi dx \ , \ \varphi \in \mathcal{D}(\Omega) \tag{6}$$

Assumptions (2) and (3) shows that u^n_ϵ exists and it is unique by the Lax-Milgram theorem. (Notice that u^n_ϵ also belongs to $W_0^{1,p}(\Omega)$). We introduce the function

$$\theta(t) = \int_0^t (1+|s|)^{p-2} ds, \ t \in \mathbb{R}$$

This kind of function has been used in [3]. We have $\theta'(t) = (1+|t|)^{p-2} \leq 1$ and $\theta(0) = 0$, therefore we have $\theta(u) \in H^1_0(\Omega)$ for every $u \in H^1_0(\Omega)$. Testing with $\theta(u^n_\epsilon)$ in (6) and using the ellipticity assumption we deduce

$$\lambda\epsilon^2 \int_\Omega (1+|u^n_\epsilon|)^{p-2} \left|\nabla_{X_1}u^n_\epsilon\right|^2 dx + \lambda \int_\Omega (1+|u^n_\epsilon|)^{p-2} \left|\nabla_{X_2}u^n_\epsilon\right|^2 dx$$
$$\leq \int_\Omega f_n \theta(u^n_\epsilon) dx \leq \frac{2}{p-1}\int_\Omega |f_n| \, (1+|u^n_\epsilon|)^{p-1} dx,$$

where we have used $|\theta(t)| \leq \frac{2(1+|t|)^{p-1}}{p-1}$. In the other hand, by Hölder's inequality we have

$$\int_\Omega \left|\nabla_{X_2}u^n_\epsilon\right|^p dx \leq \left(\int_\Omega (1+|u^n_\epsilon|)^{p-2}\left|\nabla_{X_2}u^n_\epsilon\right|^2 dx\right)^{\frac{p}{2}} \left(\int_\Omega (1+|u^n_\epsilon|)^p dx\right)^{1-\frac{p}{2}}$$

From the two previous integral inequalities we deduce

$$\int_\Omega \left|\nabla_{X_2}u^n_\epsilon\right|^p dx \leq \left(\frac{2}{\lambda(p-1)}\int_\Omega |f_n| \, (1+\left|u^n_{\epsilon_k}\right|)^{p-1} dx\right)^{\frac{p}{2}} \times$$
$$\left(\int_\Omega (1+\left|u^n_{\epsilon_k}\right|)^p dx\right)^{1-\frac{p}{2}}$$

By Hölder's inequality we get

$$\left\|\nabla_{X_2}u^n_\epsilon\right\|_{L^p(\Omega)} \leq \left(\frac{2\left\|f_n\right\|_{L^p}}{\lambda(p-1)}\right)^{\frac{1}{2}} \left(\int_\Omega (1+|u^n_\epsilon|)^p dx\right)^{\frac{1}{2p}} \tag{7}$$

Using Minkowski inequality we get

$$\left\|\nabla_{X_2}u^n_\epsilon\right\|^2_{L^p(\Omega)} \leq C(1+\left\|u^n_\epsilon\right\|_{L^p(\Omega)}),$$

Thanks to Poincaré's inequality $\|u_\epsilon^n\|_{L^p(\Omega)} \leq C_\Omega \|\nabla_{X_2} u_\epsilon^n\|_{L^p(\Omega)}$ we obtain

$$\|\nabla_{X_2} u_\epsilon^n\|_{L^p(\Omega)}^2 \leq C'(1 + \|\nabla_{X_2} u_\epsilon^n\|_{L^p(\Omega)}),$$

where the constant C' depends on p, λ, $mes(\Omega)$, M and C_Ω. Whence, we deduce

$$\|u_\epsilon^n\|_{L^p(\Omega)} \, , \; \|\nabla_{X_2} u_\epsilon^n\|_{L^p(\Omega)} \leq C'' \tag{8}$$

Similarly we obtain

$$\|\epsilon \nabla_{X_1} u_\epsilon^n\|_{L^p(\Omega)} \leq C''', \tag{9}$$

where the constants C'', C''' are independent of n and ϵ, so

$$\|u_\epsilon^n\|_{W^{1,p}(\Omega)} \leq \frac{Const}{\epsilon} \tag{10}$$

Fix ϵ, since $W^{1,p}(\Omega)$ is reflexive then (10) implies that there exists a subsequence $(u_{\epsilon_k}^{n_l(\epsilon)})_{l\in\mathbb{N}}$ and $u_\epsilon \in W_0^{1,p}(\Omega)$ such that $u_\epsilon^{n_l(\epsilon)} \rightharpoonup u_\epsilon \in W_0^{1,p}(\Omega)$ (as $l \to \infty$) in $W^{1,p}(\Omega)-$weak. Now, passing to the limit in (6) as $l \to \infty$ we deduce

$$\int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla \varphi dx = \int_\Omega f \varphi dx \; , \; \varphi \in \mathcal{D}(\Omega) \tag{11}$$

Whence u_ϵ is a weak solution of (2) ($u_\epsilon = 0$ on $\partial\Omega$ in the trace sense of $W^{1,p}-$functions, indeed the trace operator is well defined since $\partial\Omega$ is Lipschitz).

Now, from (8) and (9) we deduce

$$\|u_\epsilon\|_{L^p(\Omega)} \leq \liminf_{l\to\infty} \left\| u_\epsilon^{n_l(\epsilon)} \right\|_{L^p(\Omega)} \leq C'$$

and similarly we obtain

$$\|\epsilon \nabla_{X_1} u_\epsilon\|_{L^p(\Omega)} \, , \; \|\nabla_{X_2} u_\epsilon\|_{L^p(\Omega)} \leq C'$$

Using reflexivity and continuity of the derivation operator on $\mathcal{D}'(\Omega)$ one can extract a subsequence $(u_{\epsilon_k})_{k\in\mathbb{N}}$ such that $\nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0$, $\epsilon_k \nabla_{X_1} u_{\epsilon_k}^n \rightharpoonup 0$, $u_{\epsilon_k} \rightharpoonup u_0$ in $L^p(\Omega) - weak$. Passing to the limit in (11) we get

$$\int_\Omega A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} \varphi dx = \int_\Omega f \varphi dx \; , \; \varphi \in \mathcal{D}(\Omega) \tag{12}$$

Now, we will prove that $u_0 \in V_p$. Since $\nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0$ and $u_{\epsilon_k} \rightharpoonup u_0$ in $L^p(\Omega) - weak$ then there exists a sequence $(U_n)_{n\in\mathbb{N}} \subset conv(\{u_{\epsilon_k}\}_{k\in\mathbb{N}})$ such that $\nabla_{X_2} U_n \to \nabla_{X_2} u_0$ in $L^p(\Omega) - strong$, where $conv(\{u_{\epsilon_k}\}_{k\in\mathbb{N}})$ is the convex hull of the set $\{u_{\epsilon_k}\}_{k\in\mathbb{N}}$. Notice that we have $U_n \in W_0^{1,p}(\Omega)$ then -up to a subsequence- we have $U_n(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$, a.e $X_1 \in \Omega^1$. And we also have -up to a subsequence- $\nabla_{X_2} U_n(X_1, \cdot) \to \nabla_{X_2} u_0(X_1, .)$ in $L^p(\Omega_{X_1}) - strong$ a.e $X_1 \in \Omega^1$. Whence $u_0(X_1, .) \in W_0^{1,p}(\Omega_{X_1})$ for a.e $X_1 \in \Omega^1$, so $u_0 \in V_p$.

Finally, we will prove that u_0 is a solution of (5). Let E be a Banach space, a family of vectors $\{e_n\}_{n\in\mathbb{N}}$ in E is said to be a Banach basis or a Schauder basis of E if for every $x \in E$ there exists a family of scalars $(\alpha_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n=0}^{\infty} \alpha_n e_n$, where the series converges in the norm of E. Notice that Schauder basis does not always exist. In [11] P. Enflo has constructed a separable reflexive Banach space without Schauder basis!. However, the Sobolev space $W_0^{1,r}$ ($1 < r < \infty$) has a Schauder basis whenever the boundary of the domain is sufficiently smooth [12]. Now, we are ready to finish the proof. Let $(U_i \times V_i)_{i\in\mathbb{N}}$ be a countable covering of Ω

such that $U_i \times V_i \subset \Omega$ where $U_i \subset \mathbb{R}^q, V_i \subset \mathbb{R}^{N-q}$ are two bounded open domains, where ∂V_i is smooth (V_i are Euclidian balls for example), such a covering always exists. Now, fix $\psi \in \mathcal{D}(V_i)$ then it follows from (12) that for every $\varphi \in \mathcal{D}(U_i)$ we have

$$\int_{U_i} \varphi dX_1 \int_{V_i} A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} \psi dX_2 \ = \int_{U_i} \varphi dX_1 \int_{V_i} f \psi dX_2$$

Whence for a.e $X_1 \in U_i$ we have

$$\int_{V_i} A_{22}(X_1, \cdot) \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_2} \psi dX_2 \ = \int_{V_i} f(X_1, \cdot) \psi dX_2$$

Notice that by density we can take $\psi \in W_0^{1,p'}(V_i)$ where p' is the conjugate of p. Using the same techniques as in [8], where we use a Schauder basis of $W_0^{1,p'}(V_i)$ and a partition of the unity, one can easily obtain

$$\int_{\Omega_{X_1}} A_{22}(X_1, \cdot) \nabla_{X_2} u_0(X_1, \cdot) \cdot \nabla_{X_2} \varphi dx = \int_{\Omega_{X_1}} f(X_1, \cdot) \varphi dx, \ \varphi \in \mathcal{D}(\Omega),$$

for a.e $X_1 \in \Omega^1$. Finally, since $u_0(X_1, \cdot) \in W_0^{1,p}(\Omega_{X_1})$ (as proved above) then $u_0(X_1, \cdot)$ is a solution of (5) (Notice that Ω_{X_1} is also a Lipschitz domain so the trace operator is well defined). $\square$

2.2. **Strong convergence.** Theorem 1 will be proved in three steps. the proof is based on the use of the approximated problem (6). In the first step, we shall construct the solution of the limit problem

Step1 : Let $u_\epsilon^n \in H_0^1(\Omega)$ be the unique solution to (6), existence and uniqueness of u_ϵ^n follows from assumptions (3), (4) as mentioned previously. One have the following

Proposition 1. *Assume (3), (4) then there exists $(u_0^n)_{n \in \mathbb{N}} \subset V_2$ such that $\epsilon u_\epsilon^n \to 0$ in $L^2(\Omega)$, $u_\epsilon^n \to u_0^n$ in V_2 for every $n \in \mathbb{N}$, in particular the two convergences holds in $L^p(\Omega)$ and V_p respectively. And u_0^n is the unique weak solution in V_2 to the problem*

$$\begin{cases} \operatorname{div}_{X_2}(A_{22}(X_1, \cdot) \nabla_{X_2} u_0^n(X_1, \cdot)) = f_n(X_1, \cdot), \ X_1 \in \Omega^1 \\ u_0^n(X_1, \cdot) = 0 \ \text{on} \ \partial\Omega_{X_1} \end{cases} \tag{13}$$

Proof. This result follows from the L^2-theory (Theorem 1 in [8]), The convergences in V_p and $L^p(\Omega)$ follow from the continuous embedding $V_2 \hookrightarrow V_p$, $L^2(\Omega) \hookrightarrow L^p(\Omega)$ $(p < 2)$. $\square$

Now, we construct u_0 the solution of the limit problem (5). Testing with $\varphi = \theta(u_0^n(X_1, \cdot))$ in the weak formulation of (13) (θ is the function introduced in subsection 2.1) and estimating like in the proof of Theorem 3 we obtain as in (7)

$$\begin{aligned} & \|\nabla_{X_2} u_0^n(X_1, \cdot)\|_{L^p(\Omega_{X_1})} \\ \leq \ & \left(\frac{\|f_n(X_1, \cdot)\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)} \right)^{\frac{1}{2}} \times \left(\int_{\Omega_{X_1}} (1 + |u_0^n(X_1, \cdot)|)^p dX_2 \right)^{\frac{1}{2p}} \end{aligned} \tag{14}$$

Integrating over Ω^1 and using Cauchy-Schwaz's inequality in the right hand side we get

$$\|\nabla_{X_2}u_0^n\|_{L^p(\Omega)}^p \leq C\,\|f_n\|_{L^p(\Omega)}^{\frac{p}{2}}\left(\int_\Omega (1+|u_0^n|)^p dx\right)^{\frac{1}{2}}$$

and therefore

$$\|\nabla_{X_2}u_0^n\|_{L^p(\Omega)}^2 \leq C'(1+\|u_0^n\|_{L^p(\Omega)})$$

Using Poincaré's inequality $\|u_0^n\|_{L^p(\Omega)} \leq C_\Omega\,\|\nabla_{X_2}u_0^n\|_{L^p(\Omega)}$ (which holds since $u_0^n(X_1,\cdot)\in W_0^{1,p}(\Omega_{X_1})$ a.e $X_1\in\Omega^1$), one can obtain the estimate

$$\|u_0^n\|_{L^p(\Omega)} \leq C'' \text{ for every } n\in\mathbb{N}, \tag{15}$$

where C'' is independent of n. Now, using the linearity of the problem and (13) with the test function $\theta(u_0^n(X_1,\cdot)-u_0^m(X_1,\cdot))$, $m,n\in\mathbb{N}$ one can obtain like in (14)

$$\begin{aligned}&\|\nabla_{X_2}\left(u_0^n(X_1,\cdot)-u_0^m(X_1,\cdot)\right)\|_{L^p(\Omega_{X_1})}\\ &\leq\left(\frac{\|f_n(X_1,\cdot)-f_m(X_1,\cdot)\|_{L^p(\Omega_{X_1})}}{\lambda(p-1)}\right)^{\frac{1}{2}}\times\\ &\qquad\qquad\left(\int_{\Omega_{X_1}}(1+|u_0^n(X_1,\cdot)-u_0^m(X_1,\cdot)|)^p dX_2\right)^{\frac{1}{2p}}\end{aligned}$$

integrating over Ω^1 and using Cauchy-Schwarz and (15) yields

$$\|\nabla_{X_2}(u_0^n-u_0^m)\|_{L^p(\Omega)} \leq C\,\|f_n-f_m\|_{L^p(\Omega)}^{\frac{1}{2}},$$

where C is independent of m and n. The Poincaré's inequality shows that

$$\|u_0^n-u_0^m\|_{V_p} \leq C'\,\|f_n-f_m\|_{L^p(\Omega)}^{\frac{1}{2}}$$

Since $(f_n)_{n\in\mathbb{N}}$ is a converging sequence in $L^p(\Omega)$ then this last inequality shows that $(u_0^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V_p, consequently there exists $u_0\in V_p$ such that $u_0^n\to u_0$ in V_p. Now, passing to the limit in (6) as $\epsilon\to 0$ we get

$$\int_\Omega A_{22}\nabla_{X_2}u_0^n\cdot\nabla_{X_2}\varphi dX_2 = \int_\Omega f_n\varphi dX_2,\ \varphi\in\mathcal{D}(\Omega)$$

Passing to the limit as $n\to\infty$ we deduce

$$\int_\Omega A_{22}\nabla_{X_2}u_0\cdot\nabla_{X_2}\varphi dX_2 = \int_\Omega f\varphi dX_2,\ \varphi\in\mathcal{D}(\Omega)$$

Then it follows as proved in Theorem 3 that u_0 satisfies (5). Whence we have proved the following

Proposition 2. *Under assumption of Proposition 1 there exists $u_0\in V_p$ solution to (5) such that $u_0^n\to u_0$ in V_p where $(u_0^n)_{n\in\mathbb{N}}$ is the sequence given in Proposition 1*

Step2 : In this second step we will construct the sequence $(u_\epsilon)_{0<\epsilon\leq 1}$ solutions of (2), one can prove the following

Proposition 3. *There exists a sequence $(u_\epsilon)_{0<\epsilon\leq 1} \subset W_0^{1,p}(\Omega)$ of weak solutions to (2) such that $u_\epsilon^n \to u_\epsilon$ in $W^{1,p}(\Omega)$ for every ϵ fixed. Moreover, $u_\epsilon^n \to u_\epsilon$ in V_p and $\epsilon\nabla_{X_2} u_\epsilon^n \to \epsilon\nabla_{X_2} u_\epsilon$, uniformly in ϵ.*

Proof. Using the linearity of (6) testing with $\theta(u_\epsilon^n - u_\epsilon^m)$, $m, n \in \mathbb{N}$ we obtain as in (7)

$$\|\nabla_{X_2} u_\epsilon^n - u_\epsilon^m\|_{L^p(\Omega)} \leq \left(\frac{\|f_n - f_m\|_{L^p}}{\lambda(p-1)}\right)^{\frac{1}{2}} \left(\int_\Omega (1 + |u_\epsilon^n - u_\epsilon^m|)^p\right)^{\frac{1}{2p}}$$

And (8) gives

$$\|\nabla_{X_2}(u_\epsilon^n - u_\epsilon^m)\|_{L^p(\Omega)} \leq C \, \|f_n - f_m\|_{L^p}^{\frac{1}{2}}$$

where C is independent of ϵ and n, whence Poincaré's inequality implies

$$\|u_\epsilon^n - u_\epsilon^m\|_{V_p} \leq C' \, \|f_n - f_m\|_{L^p}^{\frac{1}{2}} \tag{16}$$

Similarly we obtain

$$\|\epsilon\nabla_{X_2}(u_\epsilon^n - u_\epsilon^m)\|_{L^p(\Omega)} \leq C'' \, \|f_n - f_m\|_{L^p}^{\frac{1}{2}} \tag{17}$$

its follows that

$$\|u_\epsilon^n - u_\epsilon^m\|_{W^{1,p}(\Omega)} \leq \frac{C}{\epsilon} \, \|f_n - f_m\|_{L^p}^{\frac{1}{2}}$$

The last inequality implies that for every ϵ fixed $(u_\epsilon^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_0^{1,p}(\Omega)$, Then there exists $u_\epsilon \in W_0^{1,p}(\Omega)$ such that $u_\epsilon^n \to u_\epsilon$ in $W^{1,p}(\Omega)$, then the passage to the limit in (6) shows that u_ϵ is a weak solution of (2). Finally (16) and (17) show that $u_\epsilon^n \to u_\epsilon$ (resp $\epsilon\nabla_{X_2} u_\epsilon^n \to \epsilon\nabla_{X_2} u_\epsilon$) in V_p (resp in $L^p(\Omega)$) uniformly in ϵ. □

Step3 : Now, we are ready to conclude. Proposition 1, 2 and 3 combined with the triangular inequality show that $u_\epsilon \to u_0$ in V_p and $\epsilon\nabla_{X_2} u_\epsilon \to 0$ in $L^p(\Omega)$, and the proof of Theorem 1 is finished.

2.3. **Convergence of the entropy solutions.** As mentioned in section 1 the entropy solution u_ϵ of (2) exists and it is unique. We shall construct this entropy solution. Using the approximated problem (6), one has a $W^{1,p}-$strongly converging sequence $u_\epsilon^n \to u_\epsilon \in W_0^{1,p}(\Omega)$ as shown in Proposition 3. We will show that $u_\epsilon \in \mathcal{T}_0^{1,2}(\Omega)$. Clearly we have$T_k(u_\epsilon^n) \in H_0^1(\Omega)$ for every $k > 0$. Now testing with $T_k(u_\epsilon^n)$ in (6) we obtain

$$\int_\Omega A_\epsilon \nabla u_\epsilon^n \cdot \nabla T_k(u_\epsilon^n) dx = \int_\Omega f_n T_k(u_\epsilon^n) dx$$

Using the ellipticity assumption we get

$$\int_\Omega |\nabla T_k(u_\epsilon^n)|^2 \leq \frac{Mk}{\lambda(1+\epsilon^2)} \tag{18}$$

Fix ϵ, k, we have $u_\epsilon^n \to u_\epsilon$ in $L^p(\Omega)$ then there exists a subsequence $(u_\epsilon^{n_l})_{l\in\mathbb{N}}$ such that $u_\epsilon^{n_l} \to u_\epsilon$ a.e $x \in \Omega$ and since T_k is bounded then it follows that $T_k(u_\epsilon^{n_l}) \to T_k(u_\epsilon)$ a.e in Ω and strongly in $L^2(\Omega)$ whence $u_\epsilon \in \mathcal{T}_0^{1,2}(\Omega)$.

It follows by (18) that there exists a subsequence still labelled $T_k(u_\epsilon^{n_l})$ such that $\nabla T_k(u_\epsilon^{n_l}) \to v_{\epsilon,k} \in L^2(\Omega)$.The continuity of ∇ on $\mathcal{D}'(\Omega)$ implies that $v_{\epsilon,k} =$

$\nabla T_k(u_\epsilon)$, whence $T_k(u_\epsilon^{n_l}) \to T_k(u_\epsilon)$ in $H^1(\Omega)$. Now, since $T_k(u_\epsilon^{n_l}) \in H_0^1(\Omega)$ then we deduce that $T_k(u_\epsilon) \in H_0^1(\Omega)$.

It follows [4] that

$$\int_\Omega A_\epsilon \nabla u_\epsilon \cdot \nabla T_k(u_\epsilon - \varphi)dx \leq \int_\Omega f T_k(u_\epsilon - \varphi)dx$$

Whence u_ϵ is the entropy solution of (2). Similarly the function u_0 (constructed in Proposition 2) is the entropy solution to (5) for a.e X_1 The uniqueness of u_0 in V_p follows from the uniqueness of the entropy solution of problem (5). Finally, the convergences given in Theorem 2 follows from Theorem 1.

Remark 1. *Uniqueness of the entropy solutions implies that it does not depend on the choice of the approximated sequence* $(f_n)_n$.

2.4. **A regularity result for the entropy solution of the limit problem.** In this subsection we assume that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded Lipschitz domains of $\mathbb{R}^q$, $\mathbb{R}^{N-q}$ respectively. We introduce the space

$$W_p = \{u \in L^p(\Omega) \mid \nabla_{X_1} u \in L^p(\Omega)\}$$

We suppose the following

$$f \in W_p \text{ and } A_{22}(x) = A_{22}(X_2) \text{ i.e } A_{22} \text{ is independent of } X_1 \tag{19}$$

Theorem 4. *Assume (3), (4), (19) then* $u_0 \in W^{1,p}(\Omega)$, *where* u_0 *is the entropy solution of (5).*

Proof. Let (u_0^n) the sequence constructed in subsection 2.2, we have $u_0^n \to u_0$ in V_p, where u_0 is the entropy solution of (5) as mentioned in the above subsection.

Let $\omega_1' \subset\subset \omega_1$ be an open subset, for $0 < h < d(\partial\omega_1, \omega_1')$ and for $X_1 \in \omega_1'$ we set $\tau_h^i u_0^n = u_0^n(X_1 + he_i, X_2)$ where $e_i = (0, .., 1, .., 0)$ then we have by (13)

$$\int_{\omega_2} A_{22} \nabla_{X_2}(\tau_h^i u_0^n - u_0^n) \cdot \nabla_{X_2} \varphi dX_2 = \int_{\omega_2} (\tau_h^i f_n - f_n)\varphi dX_2 \ , \ \ \varphi \in \mathcal{D}(\omega_2)$$

where we have used $A_{22}(x) = A_{22}(X_2)$.

We introduce the function $\theta_\delta(t) = \int\limits_0^t (\delta + |s|)^{p-2} \, ds$, $\delta > 0$, $t \in \mathbb{R}$ we have $0 < \theta_\delta'(t) = (\delta + |t|)^{p-2} \leq \delta^{p-2}$ and $|\theta_\delta(t)| \leq \frac{2(\delta+|t|)^{p-1}}{p-1}$

Testing with $\varphi = \frac{1}{h}\theta_\delta(\frac{\tau_h^i u_0^n - u_0^n}{h}) \in H_0^1(\omega_2)$. To make the notations less heavy we set

$$U = \frac{\tau_h^i u_0^n - u_0^n}{h}, \ \frac{(\tau_h^i f_n - f_n)}{h} = F$$

Then we get

$$\int_{\omega_2} \theta_\delta'(U) A_{22} \nabla_{X_2} U \cdot \nabla_{X_2} U dX_2 = \int_{\omega_2} F \theta_\delta(U) dX_2$$

Using the ellipticity assumption for the left hand side and Hölder's inequality for the right hand side of the previous inequality we deduce

$$\lambda \int_{\omega_2} \theta_\delta'(U) \left|\nabla_{X_2} U\right|^2 dX_2 \leq \frac{2}{p-1} \|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta + |U|)^p \, dX_2 \right)^{\frac{p-1}{p}}$$

Using Hölder's inequality we derive

$$\|\nabla_{X_2}U\|^p_{L^p(\omega_2)} \leq \left(\int_{\omega_2} \theta'_\delta(U)\,|\nabla_{X_2}U|^2\,dX_2\right)^{\frac{p}{2}} \left(\int_{\omega_2} \theta'_\delta(U)^{\frac{p}{p-2}}\,dX_2\right)^{\frac{2-p}{2}}$$

$$\leq \left(\frac{2}{\lambda(p-1)}\,\|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta+|U|)^p\,dX_2\right)^{\frac{p-1}{p}}\right)^{\frac{p}{2}} \times$$

$$\left(\int_{\omega_2} \theta'_\delta(U)^{\frac{p}{p-2}}\,dX_2\right)^{\frac{2-p}{2}}$$

Then we deduce

$$\|\nabla_{X_2}U\|^2_{L^p(\omega_2)} \leq \frac{2}{\lambda(p-1)}\,\|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (\delta+|U|)^p\,dX_2\right)^{\frac{1}{p}}$$

Now passing to the limit as $\delta \to 0$ using the Lebesgue theorem we deduce

$$\|\nabla_{X_2}U\|^2_{L^p(\omega_2)} \leq \frac{2}{\lambda(p-1)}\,\|F\|_{L^p(\omega_2)} \left(\int_{\omega_2} (|U|)^p\,dX_2\right)^{\frac{1}{p}},$$

and Poincaré's inequality gives

$$\|\nabla_{X_2}U\|_{L^p(\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)}\,\|F\|_{L^p(\omega_2)}$$

Now, integrating over ω'_1 yields

$$\left\|\frac{\tau^i_h u^n_0 - u^n_0}{h}\right\|_{L^p(\omega'_1\times\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\|\frac{(\tau^i_h f_n - f_n)}{h}\right\|_{L^p(\omega'_1\times\omega_2)}$$

Passing to the limit as $n \to \infty$ using the invariance of the Lebesgue measure under translations we get

$$\left\|\frac{\tau^i_h u_0 - u_0}{h}\right\|_{L^p(\omega'_1\times\omega_2)} \leq \frac{2C_{\omega_2}}{\lambda(p-1)} \left\|\frac{(\tau^i_h f - f)}{h}\right\|_{L^p(\omega'_1\times\omega_2)}$$

Whence, since $f \in W_p$ then

$$\left\|\frac{\tau^i_h u_0 - u_0}{h}\right\|_{L^p(\omega'_1\times\omega_2)} \leq C,$$

where C is independent of h, therefore we have $\nabla_{X_1}u_0 \in L^p(\Omega)$. Combining this with $u_0 \in V_p$ we get the desired result. □

3. The Rate of convergence Theorem

In this section we suppose that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 are two bounded Lipschitz domains of $\mathbb{R}^q$ and $\mathbb{R}^{N-q}$ respectively. We suppose that A_{12}, A_{22} and f depend on X_2 only i.e $A_{12}(x) = A_{12}(X_2)$, $A_{22}(x) = A_{22}(X_2)$ and $f(x) = f(X_2) \in L^p(\omega_2)$ $(1 < p < 2)$, $f \notin L^2(\omega_2)$.

Let u_ϵ, u_0 be the unique entropy solutions of (2), (5) respectively then under the above assumptions we have the following

Theorem 5. *For every $\omega'_1 \subset\subset \omega_1$ and $m \in \mathbb{N}^*$ there exists $C \geq 0$ independent of ϵ such that*

$$\|u_\epsilon - u_0\|_{W^p(\omega'_1\times\omega_2)} \leq C\epsilon^m$$

Proof. Let u_ϵ, u_0 be the entropy solutions of (2), (5) respectively, we use the approximated sequence $(u_\epsilon^n)_{\epsilon,n}$, $(u_0^n)_n$ introduced in section 2. Subtracting (13) from (6) we obtain

$$\int_\Omega A_\epsilon \nabla (u_\epsilon^n - u_0^n)\cdot \nabla\varphi dx = 0,$$

where we have used that u_0^n is independent of X_1 (since f and A_{22} are independent of X_1) and that A_{12} is independent of X_1.

Let $\omega_1' \subset\subset \omega_1$ then there exists $\omega_1' \subset\subset \omega_1'' \subset\subset \omega_1$. We introduce the function $\rho\in\mathcal{D}(\omega_1)$ such that $Supp(\rho)\subset\omega_1''$ and $\rho=1$ on ω_1'(we can choose $0\leq\rho\leq 1$) Testing with $\varphi=\rho^2\theta_\delta(u_\epsilon^n-u_0^n)\in H_0^1(\Omega)$ (we can check easily that this function belongs to $H_0^1(\Omega)$ using approximation argument) in the above integral equality we get

$$\begin{aligned}
&\int_\Omega \rho^2\theta_\delta'(u_\epsilon^n-u_0^n)A_\epsilon\nabla(u_\epsilon^n-u_0^n)\cdot\nabla(u_\epsilon^n-u_0^n)dx\\
&=-\int_\Omega \rho\theta_\delta(u_\epsilon^n-u_0^n)A_\epsilon\nabla(u_\epsilon^n-u_0^n)\cdot\nabla\rho dx\\
&=-\epsilon^2\int_\Omega \rho\theta_\delta(u_\epsilon^n-u_0^n)A_{11}\nabla_{X_1}(u_\epsilon^n-u_0^n)\cdot\nabla_{X_1}\rho dx\\
&\qquad\qquad -\epsilon\int_\Omega \rho\theta_\delta(u_\epsilon^n-u_0^n)A_{12}\nabla_{X_2}(u_\epsilon^n-u_0^n)\cdot\nabla_{X_1}\rho dx
\end{aligned}$$

where we have used that ρ is independent of X_2.

Using the ellipticity assumption for the left hand side and assumption (4) for the right hand side of previous equality we deduce

$$\begin{aligned}
&\epsilon^2\lambda\int_\Omega \theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_1}(u_\epsilon^n-u_0^n)\right|^2dx+\lambda\int_\Omega\theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_2}(u_\epsilon^n-u_0^n)\right|^2dx\\
&\leq\epsilon^2C\int_\Omega\rho\left|\theta_\delta(u_\epsilon^n-u_0^n)\right|\left|\nabla_{X_1}(u_\epsilon^n-u_0^n)\right|dx\\
&\qquad\qquad+\epsilon C\int_\Omega\rho\left|\theta_\delta(u_\epsilon^n-u_0^n)\right|\left|\nabla_{X_2}(u_\epsilon^n-u_0^n)\right|dx
\end{aligned}$$

Where $C\geq 0$ depends on A and ρ. Using Young's inequality $ab\leq\dfrac{a^2}{2c}+c\dfrac{b^2}{2}$ for the two terms in the right hand side of the previous inequality we obtain

$$\begin{aligned}
&\epsilon^2\frac{\lambda}{2}\int_\Omega \theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_1}(u_\epsilon^n-u_0^n)\right|^2dx+\frac{\lambda}{2}\int_\Omega\theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_2}(u_\epsilon^n-u_0^n)\right|^2dx\\
&\qquad\qquad\leq\epsilon^2C'\int_{\omega_1''\times\omega_2}\left|\theta_\delta(u_\epsilon^n-u_0^n)\right|^2\theta_\delta'(u_\epsilon^n-u_0^n)^{-1}dx
\end{aligned}$$

Whence

$$\begin{aligned}
&\epsilon^2\frac{\lambda}{2}\int_\Omega \theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_1}(u_\epsilon^n-u_0^n)\right|^2dx+\frac{\lambda}{2}\int_\Omega\theta_\delta'(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_2}(u_\epsilon^n-u_0^n)\right|^2dx\\
&\qquad\qquad\leq\frac{4}{(p-1)^2}\epsilon^2C'\int_{\omega_1''\times\omega_2}(\delta+|u_\epsilon^n-u_0^n|)^pdx
\end{aligned}$$

where C'' is independent of ϵ and n

Now, using Hölder's inequality and the previous inequality we deduce

$$\epsilon^2\frac{\lambda}{2}\left\|\rho\nabla_{X_1}(u_\epsilon^n-u_0^n)\right\|^2_{L^p(\Omega)}+\frac{\lambda}{2}\left\|\rho\nabla_{X_2}(u_\epsilon^n-u_0^n)\right\|^2_{L^p(\Omega)}$$
$$\leq\left[\begin{array}{l}\epsilon^2\frac{\lambda}{2}\left(\int_\Omega\theta'_\delta(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_1}(u_\epsilon^n-u_0^n)\right|^2dx\right)\\+\frac{\lambda}{2}\left(\int_\Omega\theta'_\delta(u_\epsilon^n-u_0^n)\left|\rho\nabla_{X_2}(u_\epsilon^n-u_0^n)\right|^2dx\right)\end{array}\right]\times$$
$$\left(\int_{\omega_1''\times\omega_2}(\delta+|u_\epsilon^n-u_0^n|)^pdx\right)^{\frac{2-p}{p}}$$
$$\leq\frac{4C'}{(p-1)^2}\epsilon^2\left(\int_{\omega_1''\times\omega_2}(\delta+|u_\epsilon^n-u_0^n|)^pdx\right)^{\frac{2}{p}}$$

Passing to the limit as $\delta\to 0$ using the Lebesgue theorem. Passing to the limit as $n\to\infty$ we get

$$\begin{aligned}&\epsilon^2\left\|\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}+\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}\\\leq\;&C''\epsilon^2\left\|(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1''\times\omega_2)}\end{aligned}\tag{20}$$

Using Poincaré's inequality

$$\left\|(u_\epsilon-u_0)\right\|_{L^p(\omega_1''\times\omega_2)}\leq C_{\omega_2}\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|_{L^p(\omega_1''\times\omega_2)},$$

we obtain

$$\epsilon^2\left\|\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}+\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}$$
$$\leq C''\epsilon^2\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1''\times\omega_2)}$$

Let $m\in\mathbb{N}^*$ then there exists $\omega_1'\subset\subset\omega_1''\subset\subset...\omega_1^{(m+1)}\subset\subset\omega_1$. Iterating the above inequality $m-$time we deduce

$$\epsilon^2\left\|\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}+\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}$$
$$\leq C_m\epsilon^{2m}\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1^{(m)}\times\omega_2)}$$

Now, from (20) (with ω_1' and ω_1'' replaced by $\omega_1^{(m)}$ and $\omega_1^{(m+1)}$ respectively) we deduce

$$\epsilon^2\left\|\nabla_{X_1}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}+\left\|\nabla_{X_2}(u_\epsilon-u_0)\right\|^2_{L^p(\omega_1'\times\omega_2)}$$
$$\leq C_m'\epsilon^{2(m+1)}\left\|u_\epsilon-u_0\right\|^2_{L^p(\omega_1^{(m+1)}\times\omega_2)}$$

Since $u_\epsilon\to u_0$ in $L^p(\Omega)$ then $\left\|u_\epsilon-u_0\right\|_{L^p(\Omega)}$ is bounded and therefore we obtain

$$\left\|u_\epsilon-u_0\right\|_{W^p(\omega_1'\times\omega_2)}\leq C_m''\epsilon^m$$

And the proof of the theorem is finished. □

Can one obtain a more better convergence rate? In fact, the anisotropic singular perturbation problem (2) can be seen as a problem in a cylinder becoming

unbounded. Indeed the two problems can be connected to each other via a scaling $\epsilon = \frac{1}{\ell}$ (see [5] for more details). So let us consider the problem

$$\begin{cases} -\operatorname{div}(\tilde{A}\nabla u_\ell) = f \\ u_\ell = 0 \quad \text{on } \partial\Omega_\ell \end{cases} \tag{21}$$

where $\tilde{A} = (\tilde{a}_{ij})$ is a $N \times N$ matrix such that

$$\tilde{a}_{ij} \in L^\infty(\mathbb{R}^q \times \omega_2) \tag{22}$$

$$\exists\lambda > 0 : \tilde{A}\xi \cdot \xi \geq \lambda \left|\xi\right|^2 \;\; \forall \xi \in \mathbb{R}^N \text{ for a.e } x \in \mathbb{R}^q \times \omega_2, \tag{23}$$

$\Omega_\ell = \ell\omega_1 \times \omega_2$ a bounded domain where ω_1, ω_2 are two bounded Lipschitz domain with ω_1 convex and containing 0.

We assume that $f \in L^p(\omega_2)$ $(1 < p < 2)$ and $\tilde{A}_{22}(x) = \tilde{A}_{22}(X_2)$, $\tilde{A}_{12}(x) = \tilde{A}_{12}(X_2)$.

We consider the limit problem

$$\begin{cases} -\operatorname{div}(\tilde{A}_{22}\nabla_{X_2} u_\infty) = f \\ u_\infty = 0 \quad \text{on } \partial\omega_2 \end{cases} \tag{24}$$

Then under the above assumptions we have

Theorem 6. *Let u_ℓ, u_∞ be the unique entropy solutions to (21) and (24) then for every $\alpha \in (0,1)$ there exists $C \geq 0, c > 0$ independent of ℓ such that*

$$\|\nabla(u_\ell - u_\infty)\|_{W^{1,p}(\Omega_{\alpha\ell})} \leq Ce^{-c\ell}$$

Proof. Let u_ℓ, u_∞ the unique entropy solutions to (21) and (24) respectively, and let (u_ℓ^n) and (u_∞^n) the approximation sequences (as in section 2). we have $u_\ell^n \to u_\ell$ in $W_0^{1,p}(\Omega_\ell)$ and $u_\infty^n \to u_\infty$ in $W_0^{1,p}(\omega_2)$.Subtracting the associated approximated problems to (21) and (24) and take the weak formulation we get

$$\int_{\Omega_\ell} \tilde{A}\nabla(u_\ell^n - u_\infty^n)\nabla\varphi dx = 0, \, \varphi \in \mathcal{D}(\Omega) \tag{25}$$

Where we have used that $\tilde{A}_{22}$, $\tilde{A}_{12}$, u_∞^n are independent of X_1. Now we will use the iteration technique introduced in [7], let $0 < \ell_0 \leq \ell - 1$, and let $\rho \in \mathcal{D}(\mathbb{R}^q)$ a bump function such that

$$0 \leq \rho \leq 1, \, \rho = 1 \text{ on } \ell_0\omega_1 \text{ and } \rho = 0 \text{ on } \mathbb{R}^q \diagdown (\ell_0 + 1)\omega_1, \; |\nabla_{X_1}\rho| \leq c_0$$

where c_0 is the universal constant (see [5]). Testing with $\rho^2\theta_\delta(u_\ell^n - u_\infty^n) \in H_0^1(\Omega_\ell)$ in (25) we get

$$\int_{\Omega_\ell} \rho^2\theta'_\delta(u_\ell^n - u_\infty^n)\tilde{A}\nabla(u_\ell^n - u_\infty^n) \cdot \nabla(u_\ell^n - u_\infty^n)dx + \int_{\Omega_\ell} \rho\theta_\delta(u_\ell^n - u_\infty^n)\tilde{A}\nabla(u_\ell^n - u_\infty^n) \cdot \nabla\rho dx = 0$$

Using the ellipticity assumption (23)

$$\int_{\Omega_\ell} \rho^2\theta'_\delta(u_\ell^n - u_\infty^n)\left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx \leq 2\int_{\Omega_\ell} \rho\left|\theta_\delta(u_\ell^n - u_\infty^n)\right|\left|\tilde{A}\nabla(u_\ell^n - u_\infty^n)\right|\left|\nabla\rho\right| dx$$

Notice that $\nabla \rho = 0$ on Ω_{ℓ_0}, and $\Omega_{\ell_0} \subset \Omega_{\ell_0+1}$ (since ω_1 is convex and containing 0). Then by the Cauchy-Schwaz inequality we get

$$\begin{aligned}
&\int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) \left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx \\
&\leq 2c_0 C \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} \rho \left|\theta_\delta(u_\ell^n - u_\infty^n)\right| \left|\nabla(u_\ell^n - u_\infty^n)\right| dx \\
&\leq 2c_0 C \left(\int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) \left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx \right)^{\frac{1}{2}} \times \\
&\qquad\qquad \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} \left|\theta_\delta(u_\ell^n - u_\infty^n)\right|^2 \theta'_\delta(u_\ell^n - u_\infty^n)^{-1} dx \right)^{\frac{1}{2}}
\end{aligned}$$

where we have used (22). Whence we get (since $\rho = 1$ on Ω_{ℓ_0})

$$\begin{aligned}
\int_{\Omega_{\ell_0}} \theta'_\delta(u_\ell^n - u_\infty^n) \left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx &\leq \int_{\Omega_\ell} \rho^2 \theta'_\delta(u_\ell^n - u_\infty^n) \left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx \\
&\leq \left(\frac{4c_0 C}{p-1}\right)^2 \int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx
\end{aligned}$$

From Hölder's inequality it holds that

$$\begin{aligned}
&\|\nabla(u_\ell^n - u_\infty^n)\|^2_{L^p(\Omega_{\ell_0})} \\
&\leq \left(\int_{\Omega_{\ell_0}} \theta'_\delta(u_\ell^n - u_\infty^n) \left|\nabla(u_\ell^n - u_\infty^n)\right|^2 dx \right) \left(\int_{\Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right)^{\frac{2-p}{p}} \\
&\leq \left(\frac{4c_0 C}{p-1}\right)^2 \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right) \left(\int_{\Omega_{\ell_0}} (\delta + |u_\ell^n - u_\infty^n|)^p dx \right)^{\frac{2-p}{p}}
\end{aligned}$$

Passing to the limit as $\delta \to 0$ (using the Lebesgue theorem) we get

$$\begin{aligned}
&\|\nabla(u_\ell^n - u_\infty^n)\|^2_{L^p(\Omega_{\ell_0})} \\
&\qquad \leq C_1 \left(\int_{\Omega_{\ell_0+1} \setminus \Omega_{\ell_0}} |u_\ell^n - u_\infty^n|^p \, dx \right) \times \left(\int_{\Omega_{\ell_0}} |u_\ell^n - u_\infty^n|^p \, dx \right)^{\frac{2-p}{p}},
\end{aligned}$$

where we have used $0 \leq \rho \leq 1$. Using Poincaré's inequality

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}$$

we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|^p_{L^p(\Omega_{\ell_0})} \leq C_2 \|u_\ell^n - u_\infty^n\|^p_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}$$

Using Poincaré's inequality

$$\|u_\ell^n - u_\infty^n\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})} \leq C_{\omega_2} \|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}$$

we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|^p_{L^p(\Omega_{\ell_0})} \leq C_3 \|\nabla(u_\ell^n - u_\infty^n)\|^p_{L^p(\Omega_{\ell_0+1} \setminus \Omega_{\ell_0})}$$

Whence

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0})}^p \leq \frac{C_3}{C_3+1}\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\ell_0+1})}^p$$

Let $\alpha \in (0,1)$, iterating this formula starting from $\alpha\ell$ we get

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})}^p \leq \left(\frac{C_3}{C_3+1}\right)^{[\alpha\ell]}\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell+[(1-\alpha)\ell]})}^p$$

Whence

$$\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_{\alpha\ell})} \leq ce^{-c'\ell}\|\nabla(u_\ell^n - u_\infty^n)\|_{L^p(\Omega_\ell)} \tag{26}$$

where $c, c' > 0$ are independent of ℓ and n.

Now we have to estimate the right hand side of (26). Testing with $\theta(u_\ell^n)$ in the approximated problem associated to (21) one can obtain as in subsection 2.1

$$\|\nabla u_\ell^n\|_{L^p(\Omega_\ell)} \leq C\ell^{\frac{q}{2}} \tag{27}$$

Similarly testing with $\theta(u_\infty^n)$ in the approximated problem associated to (24). we get

$$\|\nabla u_\infty^n\|_{L^p(\Omega_\ell)} \leq C'\ell^{\frac{q}{2}} \tag{28}$$

Replace (28), (27) in (26) and passing to the limit as $n \to \infty$ we obtain the desired result. $\square$

Corollary 2. *Under the above assumptions then for every $\alpha \in (0,1)$ there exists $C \geq 0$, $c > 0$ independent of ϵ such that*

$$\|u_\epsilon - u_0\|_{W^{1,p}(\alpha\omega_1\times\omega_2)} \leq Ce^{-\frac{c}{\epsilon}}$$

where u_ϵ, u_0 are the entropy solutions to (2) and (5) respectively

Remark 2. *It is very difficult to prove the rate convergence theorem for general data. When $f(x) = f_1(X_2) + f_2(x)$ with $f_1 \in L^p(\omega_2)$ and $f_2 \in W_2$ we only have the estimates*

$$\epsilon\|\nabla_{X_1}(u_\epsilon - u_0)\|_{L^p(\omega_1'\times\omega_2)} + \|\nabla_{X_2}(u_\epsilon - u_0)\|_{L^p(\omega_1'\times\omega_2)} + \|u_\epsilon - u_0\|_{L^p(\omega_1'\times\omega_2)} \leq C\epsilon$$

This follows from the linearity of the equation, Theorem 5 and the L^2–theory [8].

4. Some Extensions to nonlinear problems and applications

4.1. **A semilinear monotone problem.** We consider the semilinear problem

$$\begin{cases} -\operatorname{div}(A_\epsilon\nabla u_\epsilon) = f + a(u_\epsilon) \\ \quad u_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases} \tag{29}$$

Where the $a : \mathbb{R} \to \mathbb{R}$ is a continuous nonincreasing function which satisfies the growth condition

$$\forall x \in \mathbb{R} : |a(x)| \leq K(1+|x|),\ K \geq 0 \tag{30}$$

and $f \in L^p(\Omega)$ where $1 < p < 2$, $f \notin L^2(\Omega)$ and A is given as in Subsection 1.2. Clearly the Nemytskii operator $u \to a(u)$ maps $L^r(\Omega) \to L^r(\Omega)$ continuously for every $1 \leq r < \infty$. The passage to the limit (formally) gives the limit problem

$$\begin{cases} -\operatorname{div}_{X_2}(A_{22}(X_1,\cdot)\nabla u_0(X_1,\cdot)) = f(X_1,\cdot) + a(u_0(X_1,\cdot)) \\ u_0(X_1,\cdot) = 0 \quad \text{on } \partial\Omega_{X_1} \end{cases} \tag{31}$$

We can suppose that $a(0) = 0$. Indeed, in the general case the right hand side of (29) can be replaced by $(a(0) + f) + b(x)$ where $b(x) = a(x) - a(0)$. Clearly b is continuous nonincreasing and satisfies $|b(x)| \leq (K + |a(0)|)(1 + |x|)$.

First of all, suppose that $f \in L^2(\Omega)$,then we have the following

Proposition 4. *Assume (3), (4) and $a(0) = 0$. Let u_ϵ be the unique weak solution in $H_0^1(\Omega)$ to (29) then $\epsilon\nabla_{X_1}u_\epsilon \to 0$ in $L^2(\Omega)$ and $u_\epsilon \to u_0$ in V_2 where u_0 in the unique solution in V_2 to the limit problem (31).*

Proof. Existence of u_ϵ follows directly by a simple application of the Schauder fixed point theorem for example. The uniqueness follows form monotonicity of a and the Poincaré's inequality.

Take u_ϵ as a test function in (29) then one can obtain the estimates

$$\epsilon \left\|\nabla_{X_1}u_\epsilon\right\|_{L^2(\Omega)}, \ \left\|\nabla_{X_2}u_\epsilon\right\|_{L^2(\Omega)}, \ \left\|u_\epsilon\right\|_{L^2(\Omega)} \leq C,$$

where C is independent of ϵ, we have used that $\int_\Omega a(u_\epsilon)u_\epsilon dx \leq 0$ (thanks to monotonicity assumption and $a(0) = 0$). And we also have (thanks to assumption (30))

$$\left\|a(u_\epsilon)\right\|_{L^2(\Omega)} \leq K(|\Omega|^{\frac{1}{2}} + C)$$

so there exists $v \in L^2(\Omega)$, $u_0 \in L^2(\Omega)$, $\nabla_{X_2}u_0 \in L^2(\Omega)$ and a subsequence $(u_{\epsilon_k})_{k\in\mathbb{N}}$ such that

$$a(u_{\epsilon_k}) \to v,\ \epsilon_k\nabla_{X_1}u_{\epsilon_k} \rightharpoonup 0,\ \nabla_{X_2}u_{\epsilon_k} \rightharpoonup \nabla_{X_2}u_0,\ u_{\epsilon_k} \rightharpoonup u_0 \text{ in } L^2(\Omega)\text{-weak} \tag{32}$$

Passing to the in the weak formulation of (29) we get

$$\int_\Omega A_{22}\nabla_{X_2}u_0 \cdot \nabla_{X_2}\varphi dx = \int_\Omega f\varphi dx \ + \int_\Omega v\varphi dx,\ \varphi \in \mathcal{D}(\Omega) \tag{33}$$

Take $\varphi = u_{\epsilon_k}$ in the previous equality and passing to the limit we get

$$\int_\Omega A_{22}\nabla_{X_2}u_0 \cdot \nabla_{X_2}u_0 dx = \int_\Omega fu_0 dx \ + \int_\Omega vu_0 dx \tag{34}$$

Let us computing the quantity

$$0 \le I_k = \int_\Omega A_{\epsilon_k} \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u_0) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_{\epsilon_k} \\ \nabla_{X_2}(u_{\epsilon_k} - u_0) \end{pmatrix} dx$$
$$- \int_\Omega (a(u_{\epsilon_k}) - a(u_0))(u_{\epsilon_k} - u_0) dx$$
$$= \int_\Omega f u_{\epsilon_k} dx - \epsilon \int_\Omega A_{12} \nabla_{X_2} u_0 \cdot \nabla_{X_1} u_{\epsilon_k} dx - \epsilon \int_\Omega A_{21} \nabla_{X_1} u_{\epsilon_k} \cdot \nabla_{X_2} u_0 dx$$
$$- \int_\Omega A_{22} \nabla_{X_2} u_{\epsilon_k} \cdot \nabla_{X_2} u_0 dx - \int_\Omega A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} u_{\epsilon_k} dx$$
$$+ \int_\Omega f u_0 dx + \int_\Omega v u_0 dx + \int_\Omega a(u_0) u_{\epsilon_k} dx$$
$$+ \int_\Omega a(u_{\epsilon_k}) u_0 dx - \int_\Omega a(u_0) u_0 dx$$

(This quantity is positive thanks to the ellipticity and monotonicity assumptions).

Passing to the limit as $k \to \infty$ using (32), (33), (34) we get

$$\lim I_k = 0$$

And finally The ellipticity assumption and Poincaré's inequality show that

$$\|\epsilon_k \nabla_{X_1} u_{\epsilon_k}\|_{L^2(\Omega)}, \ \|\nabla_{X_2}(u_{\epsilon_k} - u_0)\|_{L^2(\Omega)}, \ \|u_{\epsilon_k} - u_0\|_{L^2(\Omega)} \to 0 \tag{35}$$

Whence (33) becomes

$$\int_\Omega A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} \varphi dx = \int_\Omega f \varphi dx \ + \int_\Omega a(u_0) \varphi dx, \ \varphi \in \mathcal{D}(\Omega) \tag{36}$$

$\|\nabla_{X_2}(u_{\epsilon_k} - u_0)\|_{L^2(\Omega)} \to 0$ shows that $u_0 \in V_2$, and therefore

$$\int_{\Omega_{X_1}} A_{22} \nabla_{X_2} u_0 \cdot \nabla_{X_2} \varphi dx = \int_{\Omega_{X_1}} f \varphi dx \ + \int_{\Omega_{X_1}} a(u_0) \varphi dx, \ \varphi \in \mathcal{D}(\Omega_{X_1})$$

Hence $u_0(X_1, \cdot)$ is a solution to (31). The uniqueness in $H_0^1(\Omega_{X_1})$ of the the solution of the limit problem (31) shows that u_0 is the unique function in V_2 which satisfies (36). Therefore the convergences (35) hold for the whole sequence $(u_\epsilon)_{0<\epsilon\le 1}$. □

Now, we are ready to give the main result of this subsection

Theorem 7. *Suppose that $f \in L^p(\Omega)$ where $1 < p < 2$ (we can suppose that $f \notin L^2(\Omega)$) then there exists $u_0 \in V_p$ such that $u_0(X_1, \cdot)$ is the unique entropy solution to (31) and we have $u_\epsilon \to u_0$ in V_p, $\epsilon \nabla_{X_1} u_\epsilon \to 0$ in $L^p(\Omega)$, where u_ϵ is the unique entropy solution to (29).*

Proof. We only give a sketch of the proof. Existence and uniqueness of the entropy solutions to (29) and (31) follows from the general result proved in [4]. As in proof of Theorem 2 we shall construct the entropy solution u_ϵ. we consider the approximated problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon^n) = f_n + a(u_\epsilon^n) \\ u_\epsilon^n = 0 \quad \text{on } \partial\Omega \end{cases}$$

We follows the same arguments as in section 2, where we use the above proposition and the following

$$\int_{\Omega}(a(u)-a(v)\theta(u-v)dx \leq 0$$

Which holds for every $u, v \in L^2(\Omega)$, in fact this follows from monotonicity of a and θ. ☐

4.2. **Nonlinear problem without monotonicity assumption.** Suppose that $\Omega = \omega_1 \times \omega_2$ where ω_1, ω_2 and consider the following nonlinear problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla u_\epsilon) = f + B(u_\epsilon) \\ u_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases} \tag{37}$$

Where $f \in L^p(\Omega)$, $1 < p < 2$ and $B : L^p(\Omega) \to L^p(\Omega)$ is a continuous nonlinear operator. We suppose that

$$\exists M \geq 0, \forall u \in L^p(\Omega) : \|B(u)\|_{L^p} \leq M \tag{38}$$

Proposition 5. *Assume (3), (4),and (38) then:*

1) There exists a sequence $(u_\epsilon)_{0<\epsilon\leq 1} \subset W_0^{1,p}(\Omega)$ of an entropy solutions to (37) which are also a weak solutions such that

$$\epsilon \|\nabla_{X_1} u_\epsilon\|_{L^p(\Omega)}, \ \|\nabla_{X_2} u_\epsilon\|_{L^p(\Omega)}, \ \|u_\epsilon\|_{L^p(\Omega)} \leq C_0,$$

where $C_0 \geq 0$ is independent of ϵ(the constant C_0 depends only on Ω, λ, f and M).

2) If $(u_\epsilon)_{0<\epsilon\leq 1}$ is a sequence of entropy and weak solutions to (37) then we have the above estimates.

Proof. 1) The existence of u_ϵ is based on the Schauder fixed point theorem, we define the mapping $\Gamma : L^p(\Omega) \to L^p(\Omega)$ by

$$v \in L^p(\Omega) \to \Gamma(v) = v_\epsilon \in W_0^{1,p}(\Omega)$$

where v_ϵ is the entropy solution of the linearized problem

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla v_\epsilon) = f + B(v) \\ v_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases} \tag{39}$$

Since the entropy solution is unique then Γ is well defined. we can prove easily (by using the approximation method) that Γ is continuous. As in subsection 2.1 we can obtain the estimates

$$\epsilon \|\nabla_{X_1} u_\epsilon\|_{L^p(\Omega)}, \ \|\nabla_{X_2} u_\epsilon\|_{L^p(\Omega)}, \ \|u_\epsilon\|_{L^p(\Omega)} \leq C_0$$

where C_0 is independent of ϵ and v (thanks to (38))

Now, define the subset

$$K = \left\{ u \in W_0^{1,p}(\Omega) : \epsilon \|\nabla_{X_1} u\|_{L^p(\Omega)}, \ \|\nabla_{X_2} u\|_{L^p(\Omega)}, \ \|u\|_{L^p(\Omega)} \leq C_0 \right\}$$

The subset K is convex and compact in $L^p(\Omega)$ thanks to the Sobolev compact embedding $W_0^{1,p}(\Omega) \subset L^p(\Omega)$.

The subset K is stable under Γ (since C_0 is independent of v as mentioned above). Whence Γ admits at least a fixed point $u_\epsilon \in K$, in other words u_ϵ is a weak solution to (37) which is also an entropy solution, this last assertion follows from the definition of Γ.

2) Let $(u_\epsilon)_{0<\epsilon\leq 1}$ be a sequence of entropy and weak solutions to (37) u_ϵ is the unique entropy solution to (39) with v replaced by u_ϵ and therefore we obtain the desired estimates as proved above. □

Remark 3. *In the general case the entropy solution u_ϵ of (37) is not necessarily unique.*

Now, assume that

$$f(x) = f(X_2),\ A_{22}(x) = A_{22}(X_2),\ A_{12}(x) = A_{12}(X_2) \tag{40}$$

And assume that for every $E \subset W_p$ bounded in $L^p(\Omega)$ we have

$$\overline{conv}\,\{B(E)\} \subset W_2, \tag{41}$$

where $\overline{conv}\,\{B(E)\}$ is the closed convex-hull of $B(E)$ in $L^p(\Omega)$. Assumption (41) appears strange. We shall give later some concrete examples of operators which satisfy this assumption. Let us prove the following

Theorem 8. *Assume (3), (4), (38), (40) and (41). Let $(u_\epsilon)_{0<\epsilon\leq 1} \subset W_0^{1,p}(\Omega)$ be an entropy and weak solution to (37) then for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of ϵ such that*

$$\forall\epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}$$

Proof. The proof is similar the one given in our preprint [14]. Let $(\Omega_i)_{j\in\mathbb{N}}$ an open covering of Ω such that $\overline{\Omega_j} \subset \Omega_{j+1}$. We equip the space $Z = W^{1,p}_{loc}(\Omega)$ with the topology generated by the family of seminorms $(p_j)_{j\in\mathbb{N}}$ defined by

$$p_j(u) = \|u_\epsilon\|_{W^{1,p}(\Omega_j)}$$

Equipped with this topology, Z is a separated locally convex topological vector space. We set $Y = L^p(\Omega)$ equipped with its natural topology. We define the family of the linear continuous mappings

$$\Lambda_\epsilon : Y \to Z$$

defined by: $g \in Y$, $\Lambda_\epsilon(g) = v_\epsilon$ where v_ϵ is the unique entropy solution to

$$\begin{cases} -\operatorname{div}(A_\epsilon \nabla v_\epsilon) = g \\ \ v_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases}$$

The continuity of Λ_ϵ follows immediately if we observe Λ_ϵ as a composition of $\Lambda_\epsilon : Y \to Y$ and the canonical injection $Y \to Z$

Now, we denote Z_w, Y_w the spaces Z, Y equipped with the weak topology respectively. then $\Lambda_\epsilon : Y_w \to Z_w$ is also continuous.

Consider the bounded (in Y) subset

$$E_0 = \left\{ u \in W_p \mid \|u\|_{L^p(\Omega)} \leq C_0 \right\},$$

where C_0 is the constant introduced in Proposition 5. Consider the subset $G = f + \overline{conv}\,\{B(E_0)\}$ where the closure is taken in the L^p−topology. Thanks to assumption (41) and (38) G is closed convex and bounded in Y. Now for every $g \in G$ the orbit $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in Z thanks to Remark 2. And therefore $\{\Lambda_\epsilon g\}_\epsilon$ is bounded in Z_w.

Clearly the set G is compact in Y_w. Then it follows by the Banach-Steinhaus theorem (applied on the quadruple Λ_ϵ, G, Y_w, Z_w) that there exists a bounded subset F in Z_w such that

$$\forall \epsilon: \ \Lambda_\epsilon(G) \subset F$$

The boundedness of F in Z_w implies its boundedness in Z.i.e For every $j \in \mathbb{N}$ there exists $C_j \geq 0$ independent of ϵ such that

$$\forall \epsilon : p_j(\Lambda_\epsilon(G)) \leq C_j$$

Let u_ϵ be an entropy and weak solution to (37) then we have $u_\epsilon \in E_0$ as proved in Proposition 5 then $\Lambda_\epsilon(f + B(u_\epsilon)) = u_\epsilon \in F$ for every ϵ, therefore

$$\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega_j)} \leq C_j$$

Whence for every $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} \geq 0$ independent of ϵ such that

$$\forall \epsilon : \|u_\epsilon\|_{W^{1,p}(\Omega')} \leq C_{\Omega'}$$

□

Now we are ready to prove the convergence theorem. Assume that

$$B : (L^p(\Omega), \tau_{L^p_{loc}}) \to L^p(\Omega) \text{ is continuous} \tag{42}$$

where $(L^p(\Omega), \tau_{L^p_{loc}})$ is the space $L^p(\Omega)$ equipped with the $L^p_{loc}(\Omega)$-topology. Notice that (42) implies that $B : L^p(\Omega) \to L^p(\Omega)$ is continuous. Then we have the following

Theorem 9. *Under assumptions of Theorem 8, assume in addition (42), suppose that Ω is convex, then there exists $u_0 \in V_p$ and a sequence $(u_{\epsilon_k})_{k\in\mathbb{N}}$ of entropy and weak solution to (37) such that*

$$\begin{aligned} \epsilon_k \nabla_{X_1} u_{\epsilon_k} &\rightharpoonup 0, \ \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0 \ \textit{in } L^p(\Omega) - \textit{weak} \\ \textit{and} \ \ u_{\epsilon_k} &\to u_0 \ \textit{in } L^p_{loc}(\Omega) - \textit{strong} \end{aligned}$$

Moreover u_0 satisfies in $\mathcal{D}'(\omega_2)$ the equation

$$-\operatorname{div}_{X_2}(A_{22}\nabla_{X_2} u_0(X_1, \cdot)) = f + B(u_0)(X_1, \cdot)$$

for a.e $X_1 \in \omega_1$

Proof. The estimates given in Proposition 5 show that there exists $u_0 \in L^p(\Omega)$ and a sequence $(u_{\epsilon_k})_{k\in\mathbb{N}}$ solutions to (37) such that

$$\epsilon_k \nabla_{X_1} u_{\epsilon_k} \rightharpoonup 0, \ \nabla_{X_2} u_{\epsilon_k} \rightharpoonup \nabla_{X_2} u_0 \text{ and } u_{\epsilon_k} \rightharpoonup u_0 \text{ in } L^p(\Omega) - \textit{weak} \tag{43}$$

As we have proved in Theorem 3 we have $u_0 \in V_p$. The particular difficulty is the passage to the limit in the nonlinear term. This assertion is guaranteed by Theorem 8. Indeed, since Ω is convex and Lipschitz then there an open covering $(\Omega_j)_{j\in\mathbb{N}}$, $\Omega_j \subset \Omega_{j+1}$ and $\overline{\Omega_j} \subset \Omega$ such that each Ω_j is a Lipschitz domain (Take an increasing sequence of number $0 < \beta_j < 1$ with $\lim \beta_j = 1$. Fix $x_0 \in \Omega$ and take $\Omega_j = \beta_j(\Omega - x_0) + x_0$, since Ω is convex then $\overline{\Omega_j} \subset \Omega$. The Lipschitz character is conserved since the multiplication by β_j and translations are C^∞ diffeomorphisms).

Theorem 8 shows that for every $j \in \mathbb{N}$ there exists $C_j \geq 0$ such that

$$\|u_\epsilon\|_{W^{1,p}(\Omega_j)} \leq C_{\Omega_j}$$

Since Ω_j is Lipschitz then the embedding $W^{1,p}(\Omega_j) \hookrightarrow L^p(\Omega_j)$ is compact [1] and therefore for each k there exists a subsequence $(u_{\epsilon_k^j})_k \subset L^p(\Omega_j)$ such that

$$u_{\epsilon_k^j} \mid_{\Omega_j} \rightarrow u_0 \mid_{\Omega_j}$$

By the diagonal process one can construct a sequence $(u_{\epsilon_k})_k$ such that $u_{\epsilon_k} \rightarrow u_0$ in $L^p(\Omega_j)$ for every j, in other words we have

$$u_{\epsilon_k} \rightarrow u_0 \text{ in } L^p_{loc}(\Omega) - strong \tag{44}$$

Now passing to the limit in the weak formulation of (37) we deduce

$$-\operatorname{div}_{X_2}(A_{22}\nabla_{X_2} u_0(X_1,\cdot)) = f + B(u_0)(X_1,\cdot),$$

where we have used (43) for the passage to the limit in the left hand side. For the passage to the limit in the nonlinear term we have used (44) and assumption (42). □

Example 1. *We give a concrete example of application of the above abstract analysis. Let $\Omega = \omega_1 \times \omega_2$ be a Lispchitz convex domain of $\mathbb{R}^q \times \mathbb{R}^{N-q}$ and let A be a bounded $(N-q) \times (N-q)$ matrix defined on ω_2 which satisfies the ellipticity assumption. Let us consider the integro-differential problem*

$$\begin{cases} -\operatorname{div}_{X_2}(A(X_2)\nabla_{X_2} u) = f(X_2) + \displaystyle\int_{\omega_1} h(X_1', X_1, X_2) a(u(X_1', X_2)) dX_1' \\ u(X_1,\cdot) = 0 \quad on\ \partial\omega_2 \end{cases} \tag{45}$$

where $h \in L^\infty(\omega_1 \times \Omega)$ and $f \in L^p(\omega_2)$, $1 < p < 2$, and a is a continuous real bounded function.

This equation is based on the Neutron transport equation (see for instance [10])

A solution to (45) is a function $u \in V_p$ Which satisfies (45) in $\mathcal{D}'(\omega_2)$. suppose that

$$\nabla_{X_1} h(X_1', X_1, X_2) \in L^\infty(\omega_1 \times \Omega)$$

Then we have

Theorem 10. *Under the assumptions of this example, (45) has at least a solution in V_p in the sense of $\mathcal{D}'(\omega_2)$ for a.e $X_1 \in \omega_1$*

Proof. We introduce the singular perturbation problem

$$\begin{cases} -\operatorname{div}_X(A_\epsilon \nabla u_\epsilon) = f(X_2) + \displaystyle\int_{\omega_1} h(X_1', X_1, X_2) a(u_\epsilon(X_1', X_2)) dX_1' \\ u_\epsilon = 0 \quad \text{on } \partial\Omega \end{cases}$$

where

$$A_\epsilon = \begin{pmatrix} \epsilon^2 I & 0 \\ 0 & A \end{pmatrix}$$

Clearly A_ϵ satisfies the ellipticity assumption and it is Clear that the operator

$$u \rightarrow \int_{\omega_1} h(X_1', X_1, X_2) a(u(X_1', X_2)) dX_1'$$

satisfies assumption (38).

We can prove easily that the above operator satisfies assumption (42). Indeed, let $u_n \to u$ in $L^p_{loc}(\Omega)$ then there exists a subsequence (u_{n_k}) (constructed by the diagonal process) such that $u_{n_k} \to u$ a.e in Ω. Since a is bounded then it follows by the Lebesgue theorem that

$$\int_{\omega_1} h(X_1', X_1, X_2)a(u_{n_k}(X_1', X_2))dX_1' \to \int_{\omega_1} h(X_1', X_1, X_2)a(u(X_1', X_2))dX_1',$$

in $L^p(\Omega)$. Whence by a contradiction argument we get

$$\int_{\omega_1} h(X_1', X_1, X_2)a(u_n(X_1', X_2))dX_1' \to \int_{\omega_1} h(X_1', X_1, X_2)a(u(X_1', X_2))dX_1',$$

in $L^p(\Omega)$

We can prove similarly as in [14] that (41) holds, therefore the assertion of the theorem is a simple application of theorem 9 □

Remark 4. *Notice that the compacity of the operator given in the previous example is not sufficient to prove a such result as in the L^2 theory* [10]. *This shows the importance of assumption (41) wich holds for the above operator.*

Does operator whose assumption (41) holds admit necessarily an integral representation as in (45)?.

Example 2. *We shall replace the integral by a general linear operator. Let us consider the following problem: Find $u \in V_p$ such that*

$$\begin{cases} -\operatorname{div}_{X_2}(A\nabla_{X_2}u) = f(X_2) + gP(ha(u)) \\ \quad u(X_1, \cdot) = 0 \quad \textit{on } \partial\omega_2 \end{cases} , \tag{46}$$

where a, A and f are defined as in Example 1.

We suppose that g, $h \in L^\infty(\Omega)$ with $Supp(h) \subset \Omega$ compact. Assume $\nabla_{X_1} g \in L^\infty(\Omega)$ and $P : L^p(\Omega) \to L^2(\omega_2)$ is a bounded linear operator.

When P is not compact then the operator $u \to gP(ha(u))$ is not necessarily compact, if this is the case then this operator cannot admit an integral representation.

Theorem 11. *Under the assumptions of this example there exists at least a solution $u \in V_p$ to (46) in the sense of $\mathcal{D}'(\omega_2)$ for a.e $X_1 \in \omega_1$*

Proof. Similarly, the proof is a simple application of theorem 9. □

5. Some Open questions

Problem 1. *Suppose that $\infty > p > 2$. Given $f \in L^p$ and consider (2), since $f \in L^2$then $u_\epsilon \to u_0$ in V_2. Assume that Ω and A are sufficiently regular .Can one prove that $u_\epsilon \to u_0$ in V_p?*

Problem 2. *What happens when $f \in L^1$? As mentioned in the introduction there exists a unique entropy solution to (2) which belongs to $\bigcap_{1 \le r < \frac{N}{N-1}} W_0^{1,r}(\Omega)$. Can one prove that $u_\epsilon \to u_0$ in V_r for some $1 \le r < \frac{N}{N-1}$? Can one prove at least weak convergence in L^r for some $1 < r < \frac{N}{N-1}$ as given in Theorem 4?*

References

[1] Robert A. Adams, John J. F. Fournier. Sobolev spaces, Pure and Applied Mathematics, Academic Press 2003.

[2] S. Agmon -A. Douglis -L. Nirenberg,Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure Appl. Math.,12 (1959), pp. 623–727.

[3] L. Boccardo, T. Gallouët, J. L. Vazquez. Nonlinear Elliptic Equations in Rn without Growth Restrictions on the Data, Journal of Differential Equations, Vol. 105, n 2, october 1993, p. 334-363.

[4] Ph. Bénilan, Philippe, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre, J.-L. Vazquez, An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm.Sup. Pisa Cl. Sci. (4) 22 (1995), no. 2, 241–273

[5] M. Chipot, Elliptic equations, an introductory cours.Birkhauser 2009 ISBN: 978-3764399818

[6] M. Chipot, On some anisotropic singular perturbation problems, Asymptotic Analysis, 55 (2007), p.125-144

[7] M. Chipot, K. Yeressian, Exponential rates of convergence by an iteration technique, C. R. Acad. Sci. Paris, Ser. I 346, (2008) p. 21-26.

[8] M. Chipot, S. Guesmia, On the asymptotic behaviour of elliptic, anisotropic singular perturbations problems,Com. Pur. App. Ana. 8 (1) (2009), pp. 179-193.

[9] M. Chipot, S.Guesmia, M. Sengouga. Singular perturbations of some nonlinear problems. J. Math. Sci. 176 (6), 2011, 828-843.

[10] M. Chipot, S. Guesmia, On a class of integro-differential problems. Commun. Pure Appl. Anal. 9(5), 2010, 1249–1262.

[11] P. Enflo, A counterexample to the approximation problem in Banach spaces, Acta Mathematica, vol. 130, 1973, p. 309-317.

[12] S. Fucik, O. John and J. Necas, On the existence of Schauder basis in Sobolev spaces, Comment. Math. Univ. Carolin. 13, (1972), 163-175.

[13] T. Gallouet, R. Herbin, Existence of a solution to a coupled elliptic system, Appl.Math. Letters 7(1994), 49–55.

[14] C. Ogabi , On a class of nonlinear elliptic, anisotropic singular perturbations problems. Submited. *Preprint:* https://hal.archives-ouvertes.fr/hal-01074262

[15] J. Serrin, Pathological solutions of elliptic differential equations, Ann. Sc. Norm. Sup. Pisa,18 (1964), pp. 385–387.

E-mail address: `chokri.ogabi@ac-grenoble.fr`